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Interval estimation for the two-parameter bathtub-shaped lifetime distribution based on records

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Abstract

In this paper, we study the estimation problems for the two-parameter bathtub-shaped lifetime distribution based on upper record values. Exact confidence intervals and exact joint confidence regions for the parameters are constructed. Approximate confidence intervals and regions are also discussed based on the asymptotic normality of the maximum likelihood estimators. A simulation study is done for the performance of all proposed confidence intervals and regions. Two numerical examples with real data set and simulated data, are presented to illustrate the methods proposed here.

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1. Introduction

The failure rate function is an important characteristic of a lifetime distribution and the shapes of the failure rate functions are qualitatively different. In practice, units in a population are followed from actual birth to death, a bathtub-shaped failure rate function is often seen. In recent years, some lifetime distributions with bathtub-shaped failure rate function have been investigated by several authors. For example, Bebbington et al. [5], Gurvich et al. [10], Haynatzki et al. [11], Hjorth [12], Mudholkar and Srivastava [15], Pham and Lai [17], Smith and Bain [18], Wang [19] and Xie et al. [22]. A recent account on bathtub-shaped failure rate functions can be found in the review article by Nadarajah [16].

In this paper, we discuss the two-parameter lifetime distribution with bathtub-shaped or increasing failure rate function proposed by Chen [7]. The cumulative distribution function (cdf) of this distribution is given by

$$F(x) = 1 - e^{\lambda(1-e^{x^\beta})}, \qquad x > 0, \quad \lambda, \beta > 0, \tag{1.1}$$

and hence the probability density function (pdf) is given by

$$f(x) = \lambda\beta x^{\beta-1} e^{[x^\beta + \lambda(1-e^{x^\beta})]} \qquad x > 0, \quad \lambda, \beta > 0.$$

The reliability function $R(x)$ and hazard (failure rate) function $H(x)$ of this distribution are given, respectively, by

$$R(x) = e^{\lambda(1-e^{x^\beta})}, \qquad x > 0, \quad \lambda, \beta > 0,$$

and

$$H(x) = \lambda\beta x^{\beta-1} e^{x^\beta} \qquad x > 0, \quad \lambda, \beta > 0.$$

The parameter β is the shape parameter which also affects the shape of the failure rate function. When $\beta < 1$, the failure rate function of this distribution has a bathtub shape. When $\beta \geq 1$, this distribution has an increasing failure rate (see, Chen [7] and Wu [21]).

Let $X_1, X_2, \ldots$ be a sequence of independent and identically distributed (iid) random variables with cdf $F(x)$ and pdf $f(x)$. An observation X_j is called an upper record value if its value exceeds that of all previous observations. That is, X_j is an upper record values if $X_j > X_i$ for every $i < j$. If $\{U(n), n \geq 1\}$ is defined by

$$U(1) = 1 \quad \text{and} \quad U(n) = \min\{j : j > U(n-1), X_j > X_{U(n-1)}\},$$

for $n \geq 2$, then the sequence $\{X_{U(n)}, n \geq 1\}$ provides a sequence of upper record statistics. The sequence $\{U(n), n \geq 1\}$ represents the record times.

The definition of record values was formulated by Chandler [6]. A record value or record statistic is the largest or smallest value obtained from a sequence of random variables. The theory of record values relies largely on the theory of order statistics. As mentioned by Ahsanullah and Nevzorov [3] records are very popular because they arise naturally in many fields of studies such as climatology, sports, medicine, traffic, industry and so on. Such records are memorials of their time. The annals of records reflect the progress in science and technology and enable us to study the evaluation of mankind on the basic of record achievements in various areas of its activity. For example, in industry and reliability studies, many products fail under stress. A wooden beam breaks when sufficient perpendicular force is applied to it, an electronic component ceases to function in an environment of too high temperature, and a battery dies under the stress of time. However, the precise breaking stress or failure point varies even among identical items. Hence, in such experiments, measurements may be made sequentially and only the record values are observed. Lee et al. [14] indicated that there are some situations in lifetime testing experiments in which a failure time of a product is recorded if it exceeds all preceding failure times. These recorded failure times are the upper record value sequence. As mentioned by Ahmadi and Balakrishnan [1], there is a connection between record values and minimal repair process, which is as follows. Let X be a lifetime of a component with cdf $F(x)$ and $X(m)$ denote the lifetime if m minimal repairs are allowed. Then, $X(m)$ has the same distribution as the m-th upper record derived from iid observations from $F(x)$. For more details and applications of record values, see, for example, Ahsanullah [2] and Arnold et al. [4].

The purpose of this paper is to construct the interval estimation for the parameters of the bathtub-shaped distribution based on record values. The rest of this paper is organized as follows. Section 2 provides the maximum likelihood estimators (MLEs) of the parameters β and λ, and also establishes the approximate confidence intervals and region for the parameters. Furthermore, the exact confidence intervals for the parameter β and exact joint confidence regions for the parameters β and λ are obtained by using some pivotal quantities. Section 3 conducts some simulations to study the performance of the proposed confidence intervals and regions. Section 4 discusses two numerical examples for illustration. Section 5 makes some conclusions.

2. Main Results

In this section, we will derive the approximate confidence intervals and region for the parameters based on the asymptotic normality of the MLEs. The exact confidence intervals for β and exact joint confidence regions for β and λ will also be discussed.

2.1. Maximum Likelihood Estimation. Let $X_{U(1)} < X_{U(2)} < \cdots < X_{U(m)}$ be the first m observed upper record values from two parameter bathtub-shaped lifetime distribution in (1.1). For notation simplicity, we will write X_i for $X_{U(i)}$. The likelihood function is given by (see Arnold et al. [4])

$$\begin{aligned} L(\beta,\lambda) &= f(x_m)\prod_{i=1}^{m-1}\frac{f(x_i)}{1-F(x_i)} \\ &= (\lambda\beta)^m e^{\lambda(1-e^{x_m^\beta})}\prod_{i=1}^{m} x_i^{\beta-1}e^{x_i^\beta}. \end{aligned}$$

The log-likelihood function is then

$$\begin{aligned} l(\beta,\lambda) &= \ln L(\beta,\lambda) \\ &= m\ln\lambda + m\ln\beta + \lambda(1-e^{x_m^\beta}) + (\beta-1)\sum_{i=1}^{m}\ln x_i + \sum_{i=1}^{m} x_i^\beta. \end{aligned}$$

The MLEs of (β,λ) can be obtained by solving the likelihood equations

$$\frac{\partial l(\beta,\lambda)}{\partial\beta} = \frac{m}{\beta} - \lambda x_m^\beta e^{x_m^\beta}\ln x_m + \sum_{i=1}^{m}\ln x_i + \sum_{i=1}^{m} x_i^\beta \ln x_i = 0,$$

and

$$\frac{\partial l(\beta,\lambda)}{\partial\lambda} = \frac{m}{\lambda} + (1-e^{x_m^\beta}) = 0.$$

The approximate confidence intervals and region for the unknown parameters have been discussed by some authors. See for example, Doostparast et al. [8] and Gupta and Kundu [9]. Here we will use the asymptotic normality of the MLEs to construct the confidence intervals and region for the parameters. To obtain the Fisher information matrix, we need

$$\frac{\partial^2 l(\beta,\lambda)}{\partial\beta^2} = -\frac{m}{\beta^2} - \lambda x_m^\beta (\ln x_m)^2 e^{x_m^\beta}[1+x_m^\beta] + \sum_{i=1}^m x_i^\beta (\ln x_i)^2,$$

$$\frac{\partial^2 l(\beta,\lambda)}{\partial\beta\partial\lambda} = \frac{\partial^2 l(\beta,\lambda)}{\partial\lambda\partial\beta} = -x_m^\beta(\ln x_m)e^{x_m^\beta},$$

and

$$\frac{\partial^2 l(\beta,\lambda)}{\partial\lambda^2} = -\frac{m}{\lambda^2}.$$

Under suitable regularity conditions, we know that $\sqrt{m}(\widehat{\beta}-\beta,\widehat{\lambda}-\lambda)'$ is approximately bivariate normal with mean $(0,0)$ and covariance matrix $I^{-1}(\beta,\lambda)$ evaluated at the MLEs $(\widehat{\beta},\widehat{\lambda})$, where

$$I(\beta,\lambda) = -\frac{1}{m}\begin{bmatrix} \dfrac{\partial^2 l(\beta,\lambda)}{\partial\beta^2} & \dfrac{\partial^2 l(\beta,\lambda)}{\partial\beta\partial\lambda} \\ \dfrac{\partial^2 l(\beta,\lambda)}{\partial\lambda\partial\beta} & \dfrac{\partial^2 l(\beta,\lambda)}{\partial\lambda^2} \end{bmatrix}.$$

Thus, the approximate confidence intervals for β and λ can be obtained in the usual way. Furthermore, note that $m[\widehat{\beta}-\beta,\widehat{\lambda}-\lambda]I(\widehat{\beta},\widehat{\lambda})[\widehat{\beta}-\beta,\widehat{\lambda}-\lambda]'$ is asymptotically chi-square distributed with 2 degrees of freedom. Now, using this result, the $100(1-\alpha)\%$ approximate joint confidence region for (β,λ) is given by

$$\left\{(\beta,\lambda):\ m[\widehat{\beta}-\beta,\widehat{\lambda}-\lambda]I(\widehat{\beta},\widehat{\lambda})[\widehat{\beta}-\beta,\widehat{\lambda}-\lambda]' \le \chi^2_\alpha(2)\right\},$$

where $\chi^2_\alpha(2)$ is the percentile of chi-square distribution with right-tail probability α and 2 degrees of freedom.

2.2. Exact Interval Estimations. Let $X_1 < X_2 < \cdots < X_m$ be the first m upper record values from the two-parameter bathtub-shaped lifetime distribution in (1.1). Set

$$Y_i = -\ln[1-F(X_i)] = \lambda(e^{X_i^\beta}-1), \qquad i = 1,2,\ldots,m.$$

Then, $Y_1 < Y_2 < \cdots < Y_m$ are the first m upper record values from a standard exponential distribution. Moreover, $Z_1 = Y_1$ and $Z_i = Y_i - Y_{i-1}$, for $i = 2,\ldots,m$, are iid standard exponential random variables (see Arnold et al. [4]). Hence,

$$V_j = 2\sum_{i=1}^j Z_i = 2\ Y_j$$

has a chi-square distribution with $2j$ degrees of freedom and

$$U_j = 2\sum_{i=j+1}^m Z_i = 2\ (Y_m - Y_j)$$

has a chi-square distribution with $2(m-j)$ degrees of freedom, where $j = 1,\ldots,m-1$. We can also find that U_j and V_j are independent random variables for each j. Let

$$T_j = \frac{U_j/2(m-j)}{V_j/2j} = \frac{j\ U_j}{(m-j)V_j} = \frac{j}{m-j}\left(\frac{Y_m-Y_j}{Y_j}\right). \tag{2.1}$$

It is easy to show that T_j has an F distribution with $2(m-j)$ and $2j$ degrees of freedom for $j = 1,\ldots,m-1$. Therefore, using the pivotal quantities T_j, $j = 1,\ldots,m-1$, we can provide $m-1$ confidence intervals for β. To obtain the confidence interval for β, we further need the following lemmas.

2.1. Lemma. *For any $0 < c_1 < c_2$, the function*

$$g(\beta) = \frac{e^{c_2^\beta} - 1}{e^{c_1^\beta} - 1}$$

is a strictly increasing function of β for any $\beta > 0$.

Proof. The proof of Lemma 2.1 can be found in Chen [7]. □

2.2. Lemma. *Suppose that $0 < c_1 < c_2 < \cdots < c_m$. Let*

$$T_j(\beta) = \frac{j}{m-j}\left[\frac{e^{c_m^\beta} - 1}{e^{c_j^\beta} - 1} - 1\right], \quad j = 1, \dots, m-1.$$

Then for all $j = 1, \dots, m-1$,

(a) *$T_j(\beta)$ is strictly increasing in β for any $\beta > 0$.*
(b) *For $t > 0$, the equation, $T_j(\beta) = t$ has a unique solution in $\beta > 0$.*

Proof. (a) By Lemma 2.1, it is easy to show that $T_j(\beta)$ is a strictly increasing function of β.
(b) Since the function $T_j(\beta)$ is strictly increasing in $\beta > 0$ with $\lim_{\beta\to 0} T_j(\beta) = 0$ and $\lim_{\beta\to\infty} T_j(\beta) = \infty$, then the lemma follows.

□

Let $F_{(\alpha),(\upsilon_1,\upsilon_2)}$ denote the upper α percentile of F distribution with υ_1 and υ_2 degrees of freedom. Lemma 2.2 can be used to construct $m-1$ exact confidence intervals for the shape parameter β based on the pivotal quantities $T_j(\beta)$, $j = 1, 2, \dots, m-1$. These exact confidence intervals are given in the following theorem.

2.3. Theorem. *Suppose that $X_1 < X_2 < \cdots < X_m$ be the first m observed upper record values from the two-parameter bathtub-shaped distribution. Then, for any $0 < \alpha < 1$ and for each $j = 1, 2, \dots, m-1$,*

$$\left(\varphi(X_1, \dots, X_m, F_{1-\frac{\alpha}{2}(2(m-j),2j)}), \varphi(X_1, \dots, X_m, F_{\frac{\alpha}{2}(2(m-j),2j)})\right),$$

is a $100(1-\alpha)\%$ confidence interval for β, where $\varphi(X_1, \dots, X_m, t)$ is the solution of β for the equation

$$\frac{j}{m-j}\left[\frac{e^{X_m^\beta} - 1}{e^{X_j^\beta} - 1} - 1\right] = t.$$

Proof. From (2.1), we know that the pivot

$$T_j(\beta) = \frac{j}{m-j}\left[\frac{e^{X_m^\beta} - 1}{e^{X_j^\beta} - 1} - 1\right],$$

has an F distribution with $2(m-j)$ and $2j$ degrees of freedom. Hence, the event

$$F_{1-\frac{\alpha}{2}(2(m-j),2j)} < \frac{j}{m-j}\left[\frac{e^{X_m^\beta} - 1}{e^{X_j^\beta} - 1} - 1\right] < F_{\frac{\alpha}{2}(2(m-j),2j)},$$

is equivalent to the event

$$\varphi(X_1, \dots, X_m, F_{1-\frac{\alpha}{2}(2(m-j),2j)}) < \beta < \varphi(X_1, \dots, X_m, F_{\frac{\alpha}{2}(2(m-j),2j)}).$$

This completes the proof. □

Now, let us consider another pivotal quantity to construct the confidence interval for parameter β as

$$W(\beta, m) = \frac{\frac{1}{m}\sum_{i=1}^{m} Y_i}{\left[\prod_{i=1}^{m} Y_i\right]^{1/m}} = \frac{\frac{1}{m}\sum_{i=1}^{m}(e^{x_i^\beta}-1)}{\left[\prod_{i=1}^{m}(e^{x_i^\beta}-1)\right]^{1/m}}.$$

It is easy to show that the distribution of $W(\beta, m)$ does not depend on (β, λ) and hence it provides a pivotal quantity for β. To derive the confidence interval for β based on this pivotal quantity, one need the following lemma.

2.4. Lemma. *Suppose that* $0 < c_1 < c_2 < \cdots < c_m$. *Let*

$$W(\beta, m) = \frac{\frac{1}{m}\sum_{i=1}^{m}(e^{c_i^\beta}-1)}{\left[\prod_{i=1}^{m}(e^{c_i^\beta}-1)\right]^{1/m}}.$$

Then,

(a) $W(\beta, m)$ *is strictly increasing in* β *for any* $\beta > 0$.
(b) *For* $t > 1$, *the equation,* $W(\beta, m) = t$ *has a unique solution in* $\beta > 0$.

Proof. (a) The proof can be found in Wu et al. [20].
(b) Since the function $W(\beta, m)$ is strictly increasing in $\beta > 0$ with $\lim_{\beta\to 0} W(\beta, m) = 1$ and $\lim_{\beta\to\infty} W(\beta, m) = \infty$, then the lemma follows. □

Let $W_{\alpha(m)}$ be the upper α percentile of the distribution of the pivotal quantity $W(\beta, m)$. We have the following theorem.

2.5. Theorem. *Suppose that* $X_1 < X_2 < \cdots < X_m$ *be the first* m *observed upper record values from the two-parameter bathtub-shaped distribution. Then, for any* $0 < \alpha < 1$,

$$\psi(X_1, \ldots, X_m, W_{1-\frac{\alpha}{2}(m)}) < \beta < \psi(X_1, \ldots, X_m, W_{\frac{\alpha}{2}(m)})$$

is a $100(1-\alpha)\%$ *confidence interval for* β, *where* $\psi(X_1, \ldots, X_m, t)$ *is the solution of* β *for the equation*

$$\frac{\frac{1}{m}\sum_{i=1}^{m}(e^{X_i^\beta}-1)}{\left[\prod_{i=1}^{m}(e^{X_i^\beta}-1)\right]^{1/m}} = t.$$

Proof. Note that

$$P\left(W_{1-\frac{\alpha}{2}(m)} < W(\beta, m) < W_{\frac{\alpha}{2}(m)}\right) = 1 - \alpha,$$

for any $0 < \alpha < 1$. Then, by Lemma 2.4, one can construct an exact confidence interval for β. □

It should be mentioned here that since the exact distribution of the pivotal quantity $W(\beta, m)$ is too hard to derive algebraically, we need to compute the percentiles of $W(\beta, m)$ by using Monte Carlo simulation. In Table 1, we present the upper percentiles $W_{\alpha(m)}$ of $W(\beta, m)$ for $m = 2, 3, \ldots, 20$ and various values of α, over 50000 replications.

Now, in order to derive the exact joint confidence region for (β, λ), let

$$S = U_j + V_j = 2Y_m. \tag{2.2}$$

It is easy to show that S has a chi-square distribution with $2m$ degrees of freedom. Furthermore, by Johnson et al. [13], T_j defined in (2.1) and S are independent for each j. Using the joint pivots $(S, T_1), \ldots, (S, T_{m-1})$, we can construct $m - 1$ exact joint confidence regions for (β, λ).

Table 1. Upper percentile $W_{\alpha(m)}$ of $W(\beta, m)$

	α									
m	0.995	0.005	0.99	0.01	0.975	0.025	0.95	0.05	0.90	0.10
2	1.0000	7.1897	1.0000	4.9480	1.0001	3.2949	1.0003	2.4000	1.0013	1.7691
3	1.0004	4.9643	1.0010	3.9014	1.0028	2.8395	1.0061	2.2901	1.0131	1.8074
4	1.0028	4.2341	1.0043	3.5076	1.0092	2.7647	1.0161	2.2669	1.0296	1.8387
5	1.0071	3.4388	1.0100	3.0886	1.0182	2.5200	1.0277	2.1453	1.0461	1.8156
6	1.0096	3.2875	1.0149	2.8818	1.0253	2.4163	1.0382	2.0899	1.0586	1.7836
7	1.0169	2.9389	1.0227	2.6522	1.0344	2.2674	1.0489	1.9794	1.0725	1.7498
8	1.0207	2.8689	1.0276	2.5550	1.0433	2.2014	1.0608	1.9594	1.0862	1.7358
9	1.0278	2.7457	1.0352	2.4686	1.0513	2.1332	1.0684	1.9258	1.0955	1.7185
10	1.0325	2.6099	1.0425	2.3344	1.0607	2.0542	1.0788	1.8808	1.1066	1.6924
11	1.0362	2.4933	1.0452	2.2812	1.0641	2.0258	1.0836	1.8503	1.1107	1.6784
12	1.0417	2.4335	1.0528	2.2743	1.0702	2.0070	1.0922	1.8420	1.1209	1.6655
13	1.0462	2.3429	1.0574	2.1808	1.0762	1.9588	1.0991	1.8082	1.1292	1.6606
14	1.0530	2.2814	1.0658	2.1009	1.0842	1.9174	1.1059	1.7818	1.1349	1.6489
15	1.0546	2.2565	1.0650	2.0953	1.0886	1.9038	1.1091	1.7598	1.1404	1.6315
16	1.0623	2.1736	1.0733	2.0553	1.0940	1.8935	1.1155	1.7673	1.1456	1.6357
17	1.0641	2.1703	1.0768	2.0406	1.0979	1.8751	1.1194	1.7487	1.1482	1.6220
18	1.0679	2.1394	1.0792	2.0114	1.1001	1.8594	1.1235	1.7355	1.1544	1.6173
19	1.0779	2.1080	1.0873	1.9745	1.1071	1.8248	1.1300	1.7167	1.1607	1.6043
20	1.0753	2.0807	1.0883	1.9579	1.1096	1.8176	1.1322	1.7005	1.1635	1.5874

Let $\chi^2_{\alpha(\upsilon)}$ be the upper α percentile of the χ^2 distribution with υ degrees of freedom. The following theorem provide $m-1$ exact joint confidence regions for (β, λ).

2.6. Theorem. *Suppose that $X_1 < X_2 < \cdots < X_m$ be the first m observed upper record values from the two-parameter bathtub-shaped distribution. Then, for any $j = 1, 2, \ldots, m-1$, the following inequalities determine a $100(1-\alpha)\%$ joint confidence region for (β, λ):*

$$\varphi\big(X_1, \ldots, X_m, F_{\frac{1+\sqrt{1-\alpha}}{2}(2(m-j),2j)}\big) < \beta < \varphi\big(X_1, \ldots, X_m, F_{\frac{1-\sqrt{1-\alpha}}{2}(2(m-j),2j)}\big),$$

and

$$\frac{\chi^2_{\frac{1+\sqrt{1-\alpha}}{2}(2m)}}{2(e^{X_m^\beta}-1)} < \lambda < \frac{\chi^2_{\frac{1-\sqrt{1-\alpha}}{2}(2m)}}{2(e^{X_m^\beta}-1)}.$$

where $0 < \alpha < 1$, and $\varphi(X_1, \ldots, X_m, t)$ is the solution of β for the equation

$$\frac{j}{m-j}\left[\frac{e^{X_m^\beta}-1}{e^{X_j^\beta}-1} - 1\right] = t.$$

Proof. From (2.2), we know that

$$S = 2\lambda(e^{X_m^\beta} - 1),$$

has a chi-square distribution with $2m$ degrees of freedom, and it is independent of T_j for each j. Next, for $0 < \alpha < 1$, we have

$$P\left(F_{\frac{1+\sqrt{1-\alpha}}{2}(2(m-j),2j)} < T_j < F_{\frac{1-\sqrt{1-\alpha}}{2}(2(m-j),2j)}\right) = \sqrt{1-\alpha},$$

and

$$P\left(\chi^2_{\frac{1+\sqrt{1-\alpha}}{2}(2m)} < S < \chi^2_{\frac{1-\sqrt{1-\alpha}}{2}(2m)}\right) = \sqrt{1-\alpha}.$$

From these relationships, we conclude that

$$P\left(F_{\frac{1+\sqrt{1-\alpha}}{2}(2(m-j),2j)} < T_j < F_{\frac{1-\sqrt{1-\alpha}}{2}(2(m-j),2j)}\, ,\right.$$
$$\left.\chi^2_{\frac{1+\sqrt{1-\alpha}}{2}(2m)} < S < \chi^2_{\frac{1-\sqrt{1-\alpha}}{2}(2m)}\right) = 1-\alpha,$$

or equivalently

$$P\Bigg(\varphi(X_1,\dots,X_m,F_{\frac{1+\sqrt{1-\alpha}}{2}(2(m-j),2j)}) < \beta <$$
$$\varphi(X_1,\dots,X_m,F_{\frac{1-\sqrt{1-\alpha}}{2}(2(m-j),2j)})\, , \frac{\chi^2_{\frac{1+\sqrt{1-\alpha}}{2}(2m)}}{2(e^{X_m^\beta}-1)} < \lambda < \frac{\chi^2_{\frac{1-\sqrt{1-\alpha}}{2}(2m)}}{2(e^{X_m^\beta}-1)}\Bigg)$$
$$= 1-\alpha.$$

This completes the proof. □

3. Simulation Results

In this section, we carry out a Monte Carlo simulation to study the performance of our proposed confidence intervals and regions. In this simulation, we randomly generate upper record sample $X_1, X_2, \dots, X_m$ from a two-parameter bathtub-shaped lifetime distribution with the values of parameters $(\beta,\lambda) = (0.5, 0.02)$, $(1, 0.1)$, and $(1.2, 0.05)$ and sample sizes $m = 5, 7, 10, 15$. We then compute the 95% confidence intervals and regions using Theorems 2.3, 2.5, and 2.6. We also provide the approximate joint confidence region obtained by the asymptotic normality of the MLEs. We replicate the process 5000 times. We present, in Tables 2 and 3, the average confidence lengths and confidence areas. The simulation results show that:

(1) The coverage probabilities of the exact confidence intervals for β and joint confidence regions for (β,λ) are close to the desired level of 0.95 for different parameters and sample sizes. But, the coverage probabilities of the approximate joint confidence region for (β,λ) are very low.

(2) The pivot $W(\beta,m)$ works better than the pivots $T_j(\beta)$, $j = 1,\dots,m-1$ to establish confidence interval for the parameter β. This is because the average confidence lengths based on $W(\beta,m)$ are smaller than those based on $T_j(\beta)$, $j = 1,\dots,m-1$.

(3) If we consider $m-1$ pivotal quantities $T_1(\beta),\dots,T_{m-1}(\beta)$ to establish the confidence intervals for the parameter β, We find that the pivotal quantity $T_j(\beta)$ provides the shortest confidence length when j is around $[\frac{m}{2}]$, where $[y]$ denotes the largest integer which is less than or equal to y.

(4) From Table 3, we observe that in the most of cases considered, the first joint pivot (S, T_1) provides the smallest confidence area for (β,λ). Thus, the first joint confidence region is the best exact joint confidence region.

(5) In most of the cases considered, the approximate method does not work well to establish the joint confidence region for (β,λ). It provides the low coverage probabilities. Also, the average confidence area based on the approximate method is bigger than those obtained based on the exact methods.

Table 2. The average confidence length (CL) and coverage probability (CP) of the 95% confidence interval for β

		(β,λ)=(0.5, 0.02)		(β,λ)=(1, 0.1)		(β,λ)=(1.2, 0.05)	
m	*Method*	CL	CP	CL	CP	CL	CP
5	$W(\beta,m)$	1.6223	0.947	3.7091	0.949	3.9695	0.952
	$T_1(\beta)$	6.6763	0.954	16.2102	0.948	18.1848	0.954
	$T_2(\beta)$	1.9591	0.956	5.1808	0.949	5.4815	0.946
	$T_3(\beta)$	1.9323	0.957	4.9852	0.952	5.6397	0.948
	$T_4(\beta)$	8.8491	0.951	15.1237	0.953	17.5400	0.951
7	$W(\beta,m)$	0.8227	0.953	2.0708	0.948	2.1691	0.953
	$T_1(\beta)$	5.0157	0.950	12.5267	0.952	13.2268	0.950
	$T_2(\beta)$	1.3491	0.951	3.2347	0.952	3.6846	0.943
	$T_3(\beta)$	1.0302	0.951	2.2803	0.951	2.6532	0.945
	$T_4(\beta)$	1.0676	0.950	2.6369	0.942	2.6851	0.945
	$T_5(\beta)$	1.4285	0.948	3.4875	0.953	4.0611	0.941
	$T_6(\beta)$	14.2993	0.950	21.7503	0.937	39.4215	0.948
10	$W(\beta,m)$	0.4918	0.951	1.1508	0.947	1.2793	0.957
	$T_1(\beta)$	3.9378	0.954	9.9073	0.950	10.1735	0.958
	$T_2(\beta)$	1.0393	0.943	2.4302	0.953	2.7124	0.949
	$T_3(\beta)$	0.6972	0.948	1.6400	0.950	1.8349	0.946
	$T_4(\beta)$	0.6120	0.954	1.3761	0.955	1.5010	0.947
	$T_5(\beta)$	0.6077	0.955	1.4188	0.948	1.5628	0.952
	$T_6(\beta)$	0.6669	0.946	1.5013	0.946	1.7014	0.948
	$T_7(\beta)$	0.7960	0.958	1.8017	0.950	2.0555	0.952
	$T_8(\beta)$	1.1632	0.954	2.8045	0.952	3.1162	0.955
	$T_9(\beta)$	6.7247	0.948	13.1327	0.947	15.3015	0.951
15	$W(\beta,m)$	0.3065	0.952	0.7895	0.950	0.8206	0.953
	$T_1(\beta)$	3.3448	0.950	7.8014	0.952	8.4577	0.947
	$T_2(\beta)$	0.7912	0.950	1.9128	0.941	2.0974	0.946
	$T_3(\beta)$	0.5138	0.950	1.1995	0.954	1.3766	0.948
	$T_4(\beta)$	0.4226	0.949	0.9857	0.947	1.1265	0.944
	$T_5(\beta)$	0.3960	0.946	0.9055	0.951	0.9934	0.955
	$T_6(\beta)$	0.3879	0.944	0.8782	0.947	0.9948	0.951
	$T_7(\beta)$	0.3795	0.946	0.8580	0.954	0.9915	0.951
	$T_8(\beta)$	0.3916	0.953	0.9160	0.950	1.0350	0.945
	$T_9(\beta)$	0.4071	0.952	0.9896	0.949	1.0614	0.946
	$T_{10}(\beta)$	0.4661	0.954	1.0737	0.937	1.1867	0.949
	$T_{11}(\beta)$	0.5410	0.947	1.1941	0.951	1.3402	0.948
	$T_{12}(\beta)$	0.6802	0.948	1.5637	0.944	1.7454	0.946
	$T_{13}(\beta)$	1.0401	0.949	2.5110	0.952	2.7780	0.949
	$T_{14}(\beta)$	5.5032	0.948	16.648	0.950	13.9555	0.957

Table 3. The average confidence area (CA) and coverage probability (CP) of the 95% confidence region for (β, λ) obtained by the exact and approximate methods.

		(β,λ)=(0.5, 0.02)		(β,λ)=(1, 0.1)		(β,λ)=(1.2, 0.05)	
m	$Method$	CA	CP	CA	CP	CA	CP
5	(S,T_1)	0.0103	0.951	0.1252	0.952	0.0699	0.950
	(S,T_2)	0.0156	0.958	0.1609	0.956	0.0904	0.953
	(S,T_3)	0.0226	0.960	0.2172	0.937	0.1256	0.954
	(S,T_4)	0.0360	0.957	0.2989	0.954	0.1861	0.948
	approx	0.0216	0.837	0.2041	0.896	0.1013	0.397
7	(S,T_1)	0.0067	0.955	0.0826	0.952	0.0454	0.952
	(S,T_2)	0.0090	0.948	0.1001	0.952	0.0549	0.944
	(S,T_3)	0.0119	0.951	0.1196	0.955	0.0692	0.950
	(S,T_4)	0.0158	0.951	0.1435	0.946	0.0880	0.952
	(S,T_5)	0.0223	0.956	0.1848	0.955	0.1167	0.947
	(S,T_6)	0.0336	0.949	0.2507	0.938	0.1658	0.951
	approx	0.0137	0.599	0.1376	0.596	0.0812	0.309
10	(S,T_1)	0.0046	0.954	0.0549	0.944	0.0297	0.956
	(S,T_2)	0.0056	0.950	0.0640	0.956	0.0350	0.954
	(S,T_3)	0.0068	0.955	0.0712	0.953	0.0412	0.952
	(S,T_4)	0.0082	0.954	0.0808	0.956	0.0472	0.943
	(S,T_5)	0.0099	0.954	0.0940	0.946	0.0555	0.952
	(S,T_6)	0.0123	0.950	0.1100	0.947	0.0671	0.954
	(S,T_7)	0.0158	0.956	0.1325	0.953	0.0836	0.941
	(S,T_8)	0.0218	0.949	0.1652	0.956	0.1083	0.956
	(S,T_9)	0.0312	0.951	0.2193	0.946	0.1509	0.954
	approx	0.0094	0.652	0.0952	0.666	0.0534	0.345
15	(S,T_1)	0.0030	0.953	0.0367	0.947	0.0198	0.948
	(S,T_2)	0.0035	0.951	0.0399	0.945	0.0222	0.950
	(S,T_3)	0.0040	0.952	0.0441	0.954	0.0247	0.954
	(S,T_4)	0.0045	0.952	0.0473	0.950	0.0271	0.948
	(S,T_5)	0.0051	0.950	0.0508	0.955	0.0303	0.949
	(S,T_6)	0.0056	0.946	0.0555	0.944	0.0330	0.950
	(S,T_7)	0.0065	0.944	0.0613	0.947	0.0362	0.951
	(S,T_8)	0.0075	0.956	0.0664	0.946	0.0406	0.952
	(S,T_9)	0.0087	0.951	0.0745	0.946	0.0472	0.950
	(S,T_{10})	0.0101	0.954	0.0845	0.946	0.0545	0.952
	(S,T_{11})	0.0122	0.948	0.0991	0.954	0.0649	0.946
	(S,T_{12})	0.0157	0.951	0.1180	0.944	0.0781	0.950
	(S,T_{13})	0.0209	0.946	0.1501	0.949	0.1009	0.941
	(S,T_{14})	0.0294	0.944	0.1941	0.942	0.1377	0.956
	approx	0.0057	0.586	0.0591	0.591	0.0338	0.315

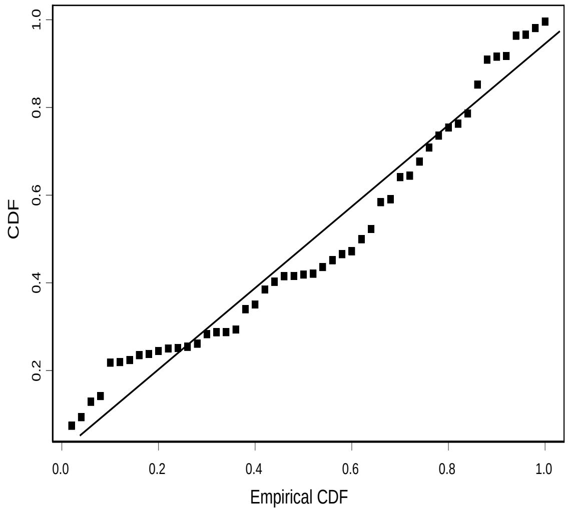

Figure 1. PP-plot of the real data set in Example 4.1.

4. Illustrative Examples

To illustrate the use of our proposed estimation methods, the following two numerical examples are discussed.

4.1. Example. (**Real data set**) Here we consider the real data of the amount of annual rainfall (in inches) recorded at the Los Angeles Civic Center for the 50 years, from 1959 to 2009. (see the website of Los Angeles Almanac: www.laalmanac.com/weather/we08aa.htm) The data are as follows:

8.180	4.850	18.790	8.380	7.930	13.680	20.440	22.000	16.580
27.470	7.740	12.320	7.170	21.260	14.920	14.350	7.210	12.300
33.440	19.670	26.980	8.960	10.710	31.280	10.430	12.820	17.860
7.660	2.480	8.081	7.350	11.990	21.000	7.360	8.110	24.350
12.440	12.400	31.010	9.090	11.570	17.940	4.420	16.420	9.250
37.960	13.190	3.210	13.530	9.080				

We check the validity of the two-parameter bathtub-shaped distribution based on the parameters $\widehat{\beta} = 0.4721$ and $\widehat{\lambda} = 0.0212$ using the Kolmogorov-Smirnov (K-S) test. It is observed that the K-S distance is 0.1385 with a corresponding p-value 0.2715. This indicates that the two-parameter bathtub-shaped distribution provides a good fit to the data. Figure 1 also shows the probability plot (PP) of the data. This figure supports our conclusion. During this period, we observe the following seven upper record values:

8.18 18.79 20.44 22.00 27.47 33.44 37.96

The MLEs of β and λ are $\widehat{\beta} = 0.432798$ and $\widehat{\lambda} = 0.0566$, respectively. Let us now obtain the approximate joint confidence region. Based on the result in Section 2.1, a

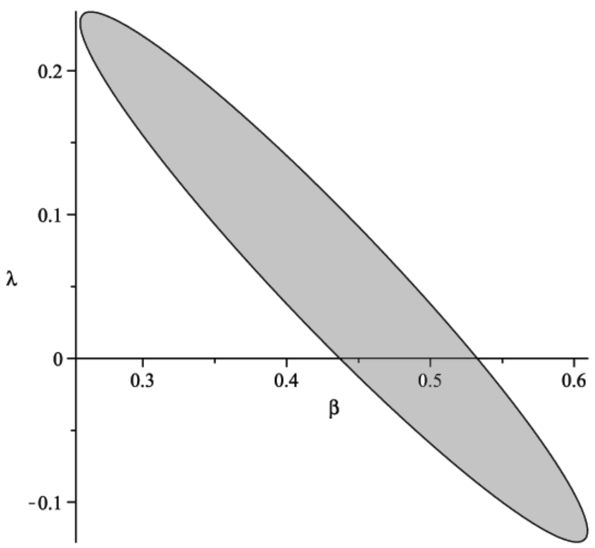

Figure 2. The 95% approximate joint confidence region in Example 4.1

95% approximate joint confidence region is the ellipse

$$A = \left\{ (\beta, \lambda) : m \begin{pmatrix} \widehat{\beta} - \beta \\ \widehat{\lambda} - \lambda \end{pmatrix}' \begin{bmatrix} 340.3978 & 312.4234 \\ 312.4234 & 311.9288 \end{bmatrix} \begin{pmatrix} \widehat{\beta} - \beta \\ \widehat{\lambda} - \lambda \end{pmatrix} - 5.9991 \leq 0 \right\},$$

where $m = 7$. This ellipse is provided in Figure 2. The area of this approximate joint confidence region is 0.0291. Now, we use the methods proposed in Section 2.2 to construct the exact confidence intervals for β and exact joint confidence region for (β, λ). To obtain the 95% confidence intervals for β, we consider the pivots $W(\beta, m)$, $T_1(\beta), \ldots, T_6(\beta)$. We need the percentiles

$$\begin{aligned}
&W_{0.025(7)} = 2.2674, \quad W_{0.975(7)} = 1.0344, \quad F_{0.025(12,2)} = 39.41462, \\
&F_{0.975(12,2)} = 0.1962375, \quad F_{0.025(10,4)} = 8.843881, \quad F_{0.975(10,4)} = 0.2237967, \\
&F_{0.025(8,6)} = 5.599623, \quad F_{0.975(8,6)} = 0.2149754, \quad F_{0.025(6,8)} = 4.651696, \\
&F_{0.975(6,8)} = 0.1785835, \quad F_{0.025(4,10)} = 4.468342, \quad F_{0.975(4,10)} = 0.1130725, \\
&F_{0.025(2,12)} = 5.095867, \quad \text{and} \quad F_{0.975(2,10)} = 0.0253713.
\end{aligned}$$

Here, the percentiles of $W_{0.025(7)}$ and $W_{0.975(7)}$ are obtained from Table 1. By Theorems 2.3 and 2.5 and using the S-PLUS package, the 95% confidence intervals and corresponding confidence lengths for β are given in Table 4.

From the simulation result in Section 3, we know that, on the average, the pivot $W(\beta, m)$ works better than the pivots $T_j(\beta)$, $j = 1, \ldots, 6$. It is not the best one in this example because the result here is based on only one sample. Among the pivots $T_j(\beta)$, $j = 1, \ldots, 6$, we observe that, in this example, the pivot $T_4(\beta)$ provides the shortest confidence interval length and hence, $(0.3723, 0.5095)$ is an optimal 95% confidence interval for β.

Table 4. The 95% confidence interval (CI) for β and corresponding confidence length (CL)

Pivot	CI	CL
$W(\beta, m)$	(0.3859,0.6861)	0.3001
$T_1(\beta)$	(0.4100,1.2874)	0.8774
$T_2(\beta)$	(0.3909,0.9163)	0.5254
$T_3(\beta)$	(0.3811,0.6094)	0.2283
$T_4(\beta)$	(0.3723,0.5095)	0.1372
$T_5(\beta)$	(0.3511,0.5321)	0.1810
$T_6(\beta)$	(0.3214, 0.6568)	0.3354

To obtain the 95% joint confidence regions for (β, λ), we need the percentiles

$$F_{0.0127(12,2)} = 78.15579, \quad F_{0.9873(12,2)} = 0.15572, \quad F_{0.0127(10,4)} = 12.79912,$$
$$F_{0.9873(10,4)} = 0.179534, \quad F_{0.0127(8,6)} = 7.37466, \quad F_{0.9873(8,6)} = 0.1699301,$$
$$F_{0.0127(6,8)} = 5.884774, \quad F_{0.9873,(6,8)} = 0.135599, \quad F_{0.0127(4,10)} = 5.569966,$$
$$F_{0.9873,(4,10)} = 0.07813, \quad F_{0.0127(2,12)} = 6.421784, \quad F_{0.9873,(2,10)} = 0.01279496,$$
$$\chi^2_{0.0127(14)} = 28.37037, \quad \text{and} \quad \chi^2_{0.9873(14)} = 4.888863.$$

By Theorem 2.6 and using the S-PLUS package for solving non-linear equation, we obtain the following 95% joint confidence regions for (β, λ) based on the joint pivots (S, T_j), $j = 1, \ldots, 6$:

$$A_1 = \left\{(\beta, \lambda) : 0.4091 < \beta < 2.1540, \quad \frac{4.888863}{2(e^{(37.96)^\beta} - 1)} < \lambda < \frac{28.37037}{2(e^{(37.96)^\beta} - 1)}\right\},$$
$$A_2 = \left\{(\beta, \lambda) : 0.3882 < \beta < 1.1574, \quad \frac{4.888863}{2(e^{(37.96)^\beta} - 1)} < \lambda < \frac{28.37037}{2(e^{(37.96)^\beta} - 1)}\right\},$$
$$A_3 = \left\{(\beta, \lambda) : 0.3792 < \beta < 0.6847, \quad \frac{4.888863}{2(e^{(37.96)^\beta} - 1)} < \lambda < \frac{28.37037}{2(e^{(37.96)^\beta} - 1)}\right\},$$
$$A_4 = \left\{(\beta, \lambda) : 0.3709 < \beta < 0.5474, \quad \frac{4.888863}{2(e^{(37.96)^\beta} - 1)} < \lambda < \frac{28.37037}{2(e^{(37.96)^\beta} - 1)}\right\},$$
$$A_5 = \left\{(\beta, \lambda) : 0.3496 < \beta < 0.5779, \quad \frac{4.888863}{2(e^{(37.96)^\beta} - 1)} < \lambda < \frac{28.37037}{2(e^{(37.96)^\beta} - 1)}\right\},$$
$$A_6 = \left\{(\beta, \lambda) : 0.3206 < \beta < 0.7445, \quad \frac{4.888863}{2(e^{(37.96)^\beta} - 1)} < \lambda < \frac{28.37037}{2(e^{(37.96)^\beta} - 1)}\right\}.$$

Figure 3 shows the above joint confidence regions for parameters β and λ. The areas of above joint confidence regions are 0.0073, 0.0109, 0.0128, 0.0145, 0.0210, and 0.0331, respectively. Thus, in this example, A_1 is the optimal joint confidence region (β, λ) since the joint pivot (S, T_1) provides the smallest confidence area. Note that the confidence areas based on all the joint pivots (S, T_j), $j = 1, \ldots, 5$ are all smaller than the area based on the approximate method. However, the area based on joint pivot (S, T_6) is larger than that based on the approximate method.

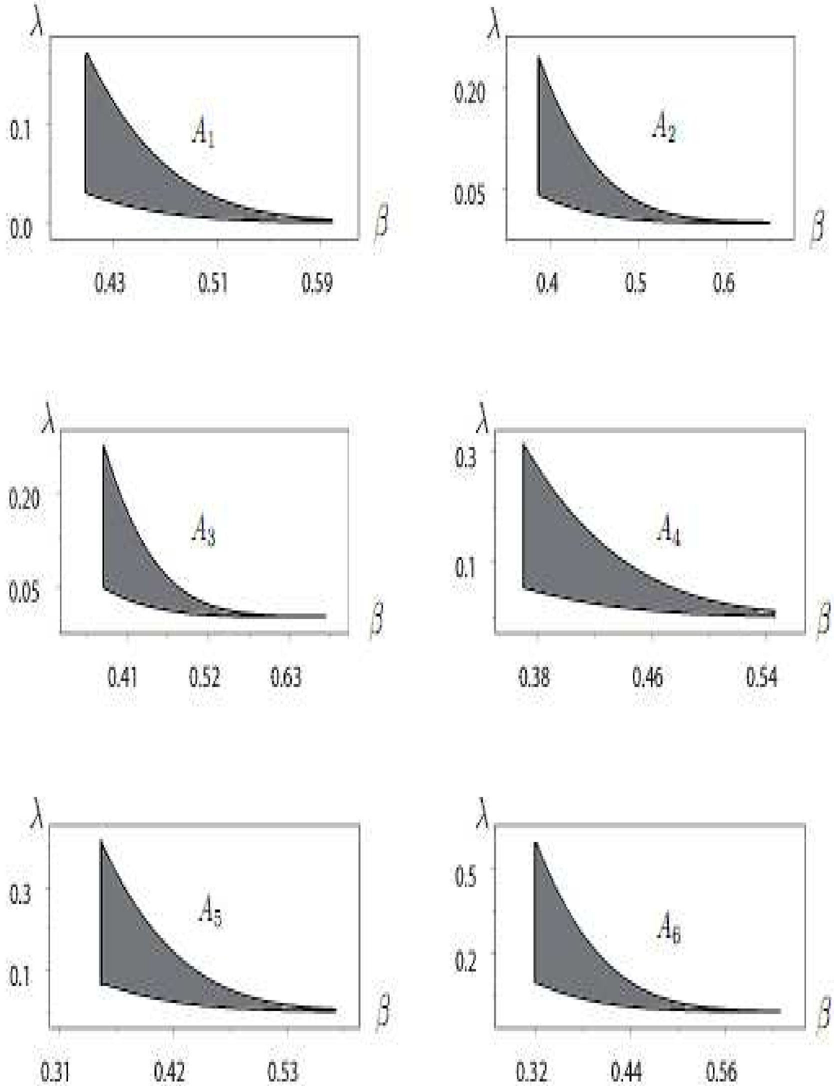

Figure 3. The 95% joint confidence region for (β, λ) in Example 4.1

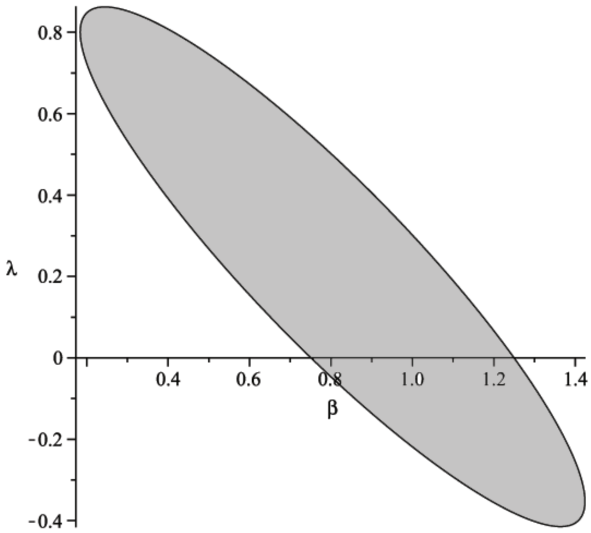

Figure 4. The 95% approximate joint confidence region in Example 4.2

4.2. Example. (**Simulated data set**) Let us consider the first four upper record values simulated from the two-parameter bathtub-shaped distribution with $\beta = 1.2$ and $\lambda = 0.05$. The simulated data are as follows:

$$1.351052 \quad 1.989847 \quad 3.030312 \quad 3.821197$$

The MLEs of β and λ are $\widehat{\beta} = 0.8039041$ and $\widehat{\lambda} = 0.2237688$, respectively. For 95% approximate joint confidence region, we have the ellipse

$$B = \left\{ (\beta, \lambda) : m \begin{pmatrix} \widehat{\beta} - \beta \\ \widehat{\lambda} - \lambda \end{pmatrix}' \begin{bmatrix} 21.19736 & 18.58491 \\ 18.58491 & 19.97105 \end{bmatrix} \begin{pmatrix} \widehat{\beta} - \beta \\ \widehat{\lambda} - \lambda \end{pmatrix} - 5.9991 \leq 0 \right\},$$

where $m = 4$. The ellipse is provided in Figure 4. The area of above joint confidence region is 0.5331. To obtain the 95% confidence intervals for β, we need the percentiles

$$W_{0.025(4)} = 2.7647, \quad W_{0.975(4)} = 1.0092, \quad F_{0.025(6,2)} = 39.33146,$$
$$F_{0.975(6,2)} = 0.137744, \quad F_{0.025(4,4)} = 9.60453, \quad F_{0.975(4,4)} = 0.1041175,$$
$$F_{0.025(2,6)} = 7.259856 \quad \text{and} \quad F_{0.975(2,6)} = 0.0254249.$$

By Theorems 2.3 and 2.5 and using the S-PLUS package, the 95% confidence intervals for β are given in Table 5.

To obtain the 95% joint confidence regions for (β, λ), we need the percentiles

$$F_{0.0127(6,2)} = 78.07254, \quad F_{0.9873(6,2)} = 0.101436, \quad F_{0.0127(4,4)} = 14.02461,$$
$$F_{0.9873(4,4)} = 0.0713032, \quad F_{0.0127(2,6)} = 9.858393, \quad F_{0.9873(2,6)} = 0.012809,$$
$$\chi^2_{0.0127(8)} = 19.4347, \quad \text{and} \quad \chi^2_{0.9873(8)} = 1.768713.$$

One can obtain the 95% joint confidence regions for (β, λ) as follows:

$$B_1 = \left\{ (\beta, \lambda) : 1.0427 < \beta < 1.8677, \quad \frac{1.768713}{2(e^{(3.821197)^\beta} - 1)} < \lambda < \frac{19.4347}{2(e^{(3.821197)^\beta} - 1)} \right\},$$

Table 5. The 95% confidence interval (CI) for β and corresponding confidence length (CL)

Pivot	CI	CL
$W(\beta, m)$	(0.9705,1.2487)	0.2782
$T_1(\beta)$	(1.0431,1.4578)	0.4147
$T_2(\beta)$	(1.0130,1.1272)	0.1142
$T_3(\beta)$	(0.9124,1.1999)	0.2875

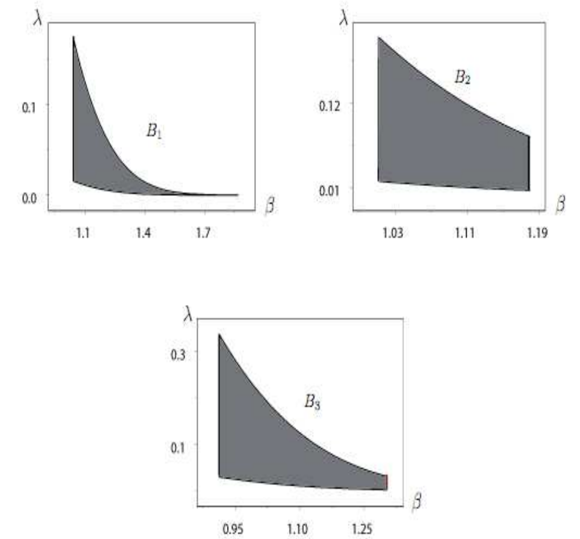

Figure 5. The 95% joint confidence region for (β, λ) in Example 4.2

$$B_2 = \left\{(\beta, \lambda) : 1.0126 < \beta < 1.1802, \quad \frac{1.768713}{2(e^{(3.821197)^\beta} - 1)} < \lambda < \frac{19.4347}{2(e^{(3.821197)^\beta} - 1)}\right\},$$

$$B_3 = \left\{(\beta, \lambda) : 0.9119 < \beta < 1.3032, \quad \frac{1.768713}{2(e^{(3.821197)^\beta} - 1)} < \lambda < \frac{19.4347}{2(e^{(3.821197)^\beta} - 1)}\right\}.$$

Figure 5 shows the above joint confidence regions. The areas of the above joint confidence regions are 0.05005, 0.0200, and 0.0238, respectively. Thus, in this example, B_2 is the optimal joint confidence regions for parameters β and λ.

5. Conclusions

The subject of record values has received attention in the past few decades. The two-parameter bathtub-shaped lifetime distribution can be widely used in reliability applications because of its failure rate function. We study the interval estimation of parameters of the two-parameter bathtub-shaped distribution based on record values. We provide three theorems based on the method of pivotal quantity to establish the exact confidence intervals and regions for the parameters. Two numerical examples are used to illustrate the proposed methods, and we also assess the confidence intervals and regions by performing a Monte Carlo simulation. The simulation results provide us some idea to choose the optimal pivots for constructing confidence intervals and regions.

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