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Research Article

Some applications on q-analog of the generalized hyperharmonic numbers of order r, $H_n^r(\alpha)$

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Abstract

In this paper, we define q-analog of the generalized harmonic numbers $H_n(\alpha)$ and the generalized hyperharmonic numbers of order r, $H_n^r(\alpha)$, and obtain some sums involving these numbers. Finally, we examine new applications of an $n \times n$ matrix $A_n = [a_{i,j}]$ with the terms $a_{i,j} = H_i^r(j,q)$.

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1. Introduction

The harmonic numbers are defined by

$$H_0 = 0, \quad H_n = \sum_{k=1}^{n} \frac{1}{k} \text{ for } n = 1, 2, \ldots.$$

These harmonic numbers are studied in various branches of number theory and combinatorial problems. There are some generalizations of the harmonic numbers H_n. The generalization of harmonic numbers, called as hyperharmonic numbers, are introduced [3, 4, 9, 12].

Benjamin et al. [3] defined the hyperharmonic numbers of order r, H_n^r, as follows: For $n, r \geq 1$,

$$H_n^r = \sum_{k=1}^{n} H_k^{r-1},$$

where for $n \geq 1$, $H_n^0 = \frac{1}{n}$ and for $r < 0$ or $n \leq 0$, $H_n^r = 0$.

From the definition of H_n^r, it is clear that

$$H_1^0 = 1 \text{ and } H_n^1 = \sum_{k=1}^{n} \frac{1}{k} = H_n.$$

The authors gave the identities

$$nH_n^r = \binom{n+r-1}{r} + rH_{n-1}^{r+1} \text{ and } H_n^r = \sum_{k=1}^{n} \binom{n+r-k-1}{r-1} \frac{1}{k}.$$

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Conway and Guy [5] obtained these numbers by taking successive partial sums of harmonic numbers. They gave the hyperharmonic numbers in terms of ordinary harmonic numbers as shown

$$H_n^r = \binom{n+r-1}{r-1}(H_{n+r-1} - H_{r-1}).$$

Considering the $n \times k$ matrix $G^r = \left[H_{n+1-i}^{r+j-1}\right]$, Bahşi and Solak [2] obtained relation between Pascal matrix and the matrix G^r, and the determinant of the $n \times n$ matrix G^r.

Using the generalized harmonic numbers of order m, $H_{n,m} = \sum\limits_{i=1}^{n} \frac{1}{i^m}$, Ömür and Koparal [11] defined two $n \times n$ matrices B_n and K_n with $b_{i,j} = H_{i,j}^r$ and $k_{i,j} = H_{i,m}^j$, respectively, where $H_{n,m}^r$ are the generalized hyperharmonic numbers of order r. They gave some new factorizations and determinants of the matrices B_n and K_n.

Definition 1.1. [6] For every ordered pair $(\alpha, n) \in \mathbb{R}^+ \times \mathbb{Z}^+$, the generalized harmonic numbers $H_n(\alpha)$ are defined by

$$H_n(\alpha) := \sum_{k=1}^{n} \frac{1}{k\alpha^k}.$$

For $\alpha = 1$, the usual harmonic numbers are $H_n(1)$. There exists integral representation in the form $H_n(\alpha) = \int\limits_{L(\alpha)}^{1} \frac{1-(1-x)^n}{x}dx$ with $L(\alpha) := 1 - \frac{1}{\alpha}$.

Ömür and Bilgin introduced the generalized hyperharmonic numbers for the generalized harmonic numbers $H_n(\alpha) = \sum\limits_{k=1}^{n} \frac{1}{k\alpha^k}$.

Definition 1.2. [10] The generalized hyperharmonic numbers of order r, $H_n^r(\alpha)$, as follows: For $r < 0$ or $n \leq 0$, $H_n^r(\alpha) = 0$ and for $n, r \geq 1$,

$$H_n^r(\alpha) = \sum_{k=1}^{n} H_k^{r-1}(\alpha),$$

where $H_n^0(\alpha) = \frac{1}{n\alpha^n}$.

They wrote the generalized hyperharmonic numbers of order r, $H_n^r(\alpha)$ as sum of the fractions $\frac{1}{k\alpha^k}$ as follows: For every ordered pair $(\alpha, r) \in \mathbb{R}^+ \times \mathbb{Z}^+$, then

$$H_n^r(\alpha) = \sum_{k=1}^{n} \binom{n+r-k-1}{r-1} \frac{1}{k\alpha^k}.$$

The $q-$analog of positive integer n is defined as

$$n_q = [n]_q := \sum_{k=0}^{n-1} q^k = \frac{1-q^n}{1-q},$$

where $q \neq 1$. Also $[0]_q = 1$. Let n and m denote integers. Then the $q-$binomial coefficients are defined by

$$\begin{bmatrix} n \\ m \end{bmatrix}_q = \begin{cases} \frac{n_q!}{m_q!(n-m)_q!} & \text{if } 0 \leq m \leq n\ , \\ 0 & \text{otherwise,} \end{cases}$$

where $n_q! = 1_q 2_q ... n_q$.

The $q-$analogs of the well-known binomial identities are given as follows ([1, 7, 13]):

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \begin{bmatrix} n-1 \\ k \end{bmatrix}_q + q^{n-k} \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}_q,$$

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}_q + q^k \begin{bmatrix} n-1 \\ k \end{bmatrix}_q ,$$

$$\sum_{k=p}^{n-m} q^{(k-p)(m+1)} \begin{bmatrix} k \\ p \end{bmatrix}_q \begin{bmatrix} n-k \\ m \end{bmatrix}_q = \begin{bmatrix} n+1 \\ p+m+1 \end{bmatrix}_q , \tag{1.1}$$

and q-analog of Vandermonde identity is

$$\sum_k q^{k(m+k-t)} \begin{bmatrix} m \\ t-k \end{bmatrix}_q \begin{bmatrix} n \\ k \end{bmatrix}_q = \begin{bmatrix} m+n \\ t \end{bmatrix}_q , \tag{1.2}$$

where $\max(0, t-m) \le k \le \min(n, t)$.

Two q-analogs of $H_n(\alpha)$ are given by, for $1 \le n$,

$$H_n(\alpha, q) := \sum_{k=1}^{n} \frac{1}{k_q \alpha^{k_q}},$$

and

$$\widetilde{H}_n(\alpha, q) := \sum_{k=1}^{n} \frac{q^k}{k_q \alpha^{k_{1/q}}}.$$

By means of $q-$difference operator, Kızılateş and Tuğlu[8] derived $q-$analogue for several well known results for harmonic numbers and gave some identities concerning $q-$hyperharmonic numbers. For example,

$$\sum_{k=m}^{n-1} q^{k-m} \begin{bmatrix} k \\ m \end{bmatrix}_q \widetilde{H}_k(1, q) = \begin{bmatrix} n \\ m+1 \end{bmatrix}_q \left(\widetilde{H}_n(1, q) - \frac{q^{m+1}}{(m+1)_q} \right).$$

Mansour and Shattuck [9] defined the q-analog of H_n^r and gave some sums related to these numbers. For example, let n and r be positive integers. For $0 \le s \le r-1$,

$$H_n^r(1, q) = \sum_{k=1}^{n} q^{k(r-s)} \begin{bmatrix} n+r-s-k-1 \\ r-s-1 \end{bmatrix}_q H_k^s(1, q),$$

and for $0 \le s \le r$,

$$H_n^r(1, q) = \sum_{k=0}^{s} q^{k(n-s+k)} \begin{bmatrix} s \\ k \end{bmatrix}_q H_{n-s+k}^{r-k}(1, q).$$

2. A q-analog of the generalized hyperharmonic numbers of order r

In this section, firstly, we will give the definition of q-analog of the generalized hyperharmonic numbers of order r, $H_n^r(\alpha)$.

Definition 2.1. For $r < 0$ or $n \le 0$, $H_n^r(\alpha, q) = 0$ and for $n, r \ge 1$, we write

$$H_n^r(\alpha, q) = \sum_{k=1}^{n} q^k H_k^{r-1}(\alpha, q), \tag{2.1}$$

where $H_n^0(\alpha, q) = \frac{q^{-1}}{n_q \alpha^{n_q}}$.

It is clear that for $n, r \geqslant 1$,

$$H_n^r(\alpha, q) = q^n H_n^{r-1}(\alpha, q) + H_{n-1}^r(\alpha, q). \tag{2.2}$$

Theorem 2.2. *For $n \ge 1$, $H_n(\alpha, q) = q\widetilde{H}_n(\alpha, 1/q)$.*

Proof. From the q-analog of the generalized harmonic numbers $H_n(\alpha,q)$, we have

$$H_n(\alpha,q)=\frac{1}{1_q\alpha^{1_q}}+\frac{1}{2_q\alpha^{2_q}}+...+\frac{1}{n_q\alpha^{n_q}},$$

and by the definition of n_q,

$$\begin{aligned}&H_n(\alpha,q)\\=&\ \frac{1}{1\alpha^1}+\frac{1}{(1+q)\alpha^{(1+q)}}+...+\frac{1}{(1+q+q^2+...+q^{n-1})\alpha^{(1+q+q^2+...+q^{n-1})}}.\end{aligned}$$

Multiplying both sides by $1/q$ and using $n_{1/q}$, we get

$$\begin{aligned}\frac{1}{q}H_n(\alpha,q)&=\frac{\frac{1}{q}}{1\alpha^1}+\frac{\frac{1}{q^2}}{\left(1+\frac{1}{q}\right)\alpha^{1+q}}+\frac{\frac{1}{q^3}}{\left(1+\frac{1}{q}+\frac{1}{q^2}\right)\alpha^{(1+q+q^2)}}+...\\&\quad+\frac{\frac{1}{q^n}}{\left(1+\frac{1}{q}+\frac{1}{q^2}+...+\frac{1}{q^{n-1}}\right)\alpha^{(1+q+q^2+...+q^{n-1})}}\\&=\sum_{k=1}^{n}\frac{1/q^k}{k_{1/q}\alpha^{k_q}}=\widetilde{H}_n(\alpha,1/q).\end{aligned}$$

Thus, the proof is complete. □

Lemma 2.3. *Let m and r be positive integers such that $0<m\le r-1$. For $n\geqslant 0$,*

$$\sum_{k=0}^{r-m-1}q^{k(n+1)}\begin{bmatrix}n+r-k-1\\r-k-1\end{bmatrix}_q=\begin{bmatrix}n+r\\n+1\end{bmatrix}_q-q^{(n+1)(r-m)}\begin{bmatrix}n+m\\n+1\end{bmatrix}_q.$$

Proof. Consider

$$\begin{aligned}&\sum_{k=0}^{r-m-1}q^{k(n+1)}\begin{bmatrix}n+r-k-1\\r-k-1\end{bmatrix}_q\\=&\sum_{k=0}^{r-1}q^{k(n+1)}\begin{bmatrix}n+r-k-1\\r-k-1\end{bmatrix}_q-q^{(r-m)(n+1)}\sum_{k=0}^{m-1}q^{k(n+1)}\begin{bmatrix}n+m-k-1\\m-k-1\end{bmatrix}_q.\end{aligned}$$

By (1.1), we have the sum as follows:

$$\sum_{k=0}^{t-1}q^{k(n+1)}\begin{bmatrix}n+t-k-1\\t-k-1\end{bmatrix}_q=\begin{bmatrix}n+t\\n+1\end{bmatrix}_q. \tag{2.3}$$

Thus, taking $t=r,m$ in (2.3), resp., we write

$$\sum_{k=0}^{r-m-1}q^{k(n+1)}\begin{bmatrix}n+r-k-1\\r-k-1\end{bmatrix}_q=\begin{bmatrix}n+r\\n+1\end{bmatrix}_q-q^{(n+1)(r-m)}\begin{bmatrix}n+m\\n+1\end{bmatrix}_q,$$

as the claim result. □

Now, with the help of (1.1), we can give the following result.

Lemma 2.4. *Let m and r be positive integers such that $0<m\le r-1$. For $n\geqslant 0$,*

$$\sum_{k=0}^{n}q^{k(m-r)}\begin{bmatrix}r+k-m-1\\k\end{bmatrix}_q=q^{n(m-r)}\begin{bmatrix}n+r-m\\n\end{bmatrix}_q. \tag{2.4}$$

Theorem 2.5. *For $n,r\ge 1$ we have*

$$\sum_{k=1}^{r}q^{n(r-k)}H_{n-1}^{k}(\alpha,q)=H_n^r(\alpha,q)-\frac{q^{nr-1}}{n_q\alpha^{n_q}}.$$

Proof. We prove this by induction on r. For $r=1$,

$$\sum_{k=1}^{1} q^{n(1-k)} H_{n-1}^{k}(\alpha,q) = H_{n-1}^{1}(\alpha,q).$$

By (2.2), we have

$$\sum_{k=1}^{1} q^{n(1-k)} H_{n-1}^{k}(\alpha,q) = H_{n}^{1}(\alpha,q) - \frac{q^{n-1}}{n_q \alpha^{n_q}}.$$

Now assume that the claim is true for $r-1$. Thus we show that the claim is true for r. Then

$$\begin{aligned}\sum_{k=1}^{r} q^{n(r-k)} H_{n-1}^{k}(\alpha,q) &= H_{n-1}^{r}(\alpha,q) + \sum_{k=1}^{r-1} q^{n(r-k)} H_{n-1}^{k}(\alpha,q)\\ &= H_{n-1}^{r}(\alpha,q) + q^{n} \sum_{k=1}^{r-1} q^{n(r-1-k)} H_{n-1}^{k}(\alpha,q).\end{aligned}$$

By the induction hypothesis and (2.2), we get

$$\begin{aligned}\sum_{k=1}^{r} q^{n(r-k)} H_{n-1}^{k}(\alpha,q) &= H_{n-1}^{r}(\alpha,q) + q^{n}\left(H_{n}^{r-1}(\alpha,q) - \frac{q^{n(r-1)-1}}{n_q \alpha^{n_q}}\right)\\ &= H_{n}^{r}(\alpha,q) - \frac{q^{nr-1}}{n_q \alpha^{n_q}}.\end{aligned}$$

The desired result is obtained. □

Theorem 2.6. *For every ordered pair $(\alpha,r)\in\mathbb{R}^{+}\times\mathbb{Z}^{+}$, we have*

$$H_{n}^{r}(\alpha,q) = \sum_{k=1}^{n} \begin{bmatrix} n+r-k-1 \\ r-1 \end{bmatrix}_q \frac{q^{rk-1}}{k_q \alpha^{k_q}}. \tag{2.5}$$

Proof. We prove this by induction on n. For $n=1$, by (2.1), we have

$$\begin{aligned}H_{1}^{r}(\alpha,q) &= \sum_{k=1}^{1} q^{k} H_{k}^{r-1}(\alpha,q) = q H_{1}^{r-1}(\alpha,q)\\ &= q\sum_{k=1}^{1} q^{k} H_{k}^{r-2}(\alpha,q) = q^{2} H_{1}^{r-2}(\alpha,q)\\ &= \ldots = q^{r} H_{1}^{0}(\alpha,q) = q^{r-1}\frac{1}{1_q \alpha^{1_q}}\\ &= \frac{q^{r-1}}{\alpha} = \begin{bmatrix} r-1 \\ r-1 \end{bmatrix}_q \frac{q^{r-1}}{1_q \alpha^{1_q}} = \sum_{k=1}^{1} \begin{bmatrix} r-k \\ r-1 \end{bmatrix}_q \frac{q^{rk-1}}{k_q \alpha^{k_q}}\\ &= \sum_{k=1}^{1} \begin{bmatrix} 1+r-k-1 \\ r-1 \end{bmatrix}_q \frac{q^{rk-1}}{k_q \alpha^{k_q}}.\end{aligned}$$

Assume that the claim is true for $n-1$. Thus we show that the claim is true for n. By Theorem 2.5, we have

$$H_{n}^{r}(\alpha,q) = \sum_{k=1}^{r} q^{n(r-k)} H_{n-1}^{k}(\alpha,q) + \frac{q^{rn-1}}{n_q \alpha^{n_q}}.$$

By the induction hypothesis, we get

$$\begin{aligned} &H_n^r(\alpha,q) \\ =& \sum_{k=1}^{r} q^{(r-k)n}\left(\sum_{t=1}^{n-1}\begin{bmatrix} n+k-t-2 \\ k-1\end{bmatrix}_q \frac{q^{kt-1}}{t_q\alpha^{t_q}}\right)+\frac{q^{rn-1}}{n_q\alpha^{n_q}} \\ =& \sum_{t=1}^{n-1}\frac{1}{t_q\alpha^{t_q}}\left(\sum_{k=0}^{r-1}\begin{bmatrix} n+k-t-1 \\ k\end{bmatrix}_q q^{t(k+1)-1+n(r-k-1)}\right)+\frac{q^{rn-1}}{n_q\alpha^{n_q}}. \end{aligned}$$

From (2.4), we have

$$\begin{aligned} H_n^r(\alpha,q) &= \sum_{t=1}^{n-1}\begin{bmatrix} n+r-t-1 \\ r-1\end{bmatrix}_q \frac{q^{rt-1}}{t_q\alpha^{t_q}}+\frac{q^{rn-1}}{n_q\alpha^{n_q}} \\ &= \sum_{t=1}^{n}\begin{bmatrix} n+r-t-1 \\ r-1\end{bmatrix}_q \frac{q^{rt-1}}{t_q\alpha^{t_q}}, \end{aligned}$$

as claimed. □

Theorem 2.7. *For $n \geq 1$ and $1 \leqslant m \leq r$, we have*

$$H_n^r(\alpha,q)=\sum_{k=0}^{n-1} q^{m(n-k)}\begin{bmatrix} k+m-1 \\ m-1\end{bmatrix}_q H_{n-k}^{r-m}(\alpha,q).$$

Proof. From (2.5), we have

$$H_n^r(\alpha,q)=\sum_{k=1}^{n}\begin{bmatrix} n+r-k-1 \\ r-1\end{bmatrix}_q \frac{q^{rk-1}}{k_q\alpha^{k_q}}.$$

Taking $r-m-1$, $m-1$, $n-k+r-2$ instead of p, m, n in (1.1), resp., we write

$$\begin{aligned} &H_n^r(\alpha,q) \\ =& \sum_{k=1}^{n}\left(\sum_{i=0}^{n-k} q^{mi}\begin{bmatrix} n-i-k+m-1 \\ m-1\end{bmatrix}_q\begin{bmatrix} i+r-m-1 \\ r-m-1\end{bmatrix}_q\right)\frac{q^{rk-1}}{k_q\alpha^{k_q}} \\ =& \sum_{k=1}^{n}\left(\sum_{i=k}^{n} q^{mi}\begin{bmatrix} n-i+m-1 \\ m-1\end{bmatrix}_q\begin{bmatrix} i-k+r-m-1 \\ r-m-1\end{bmatrix}_q\right)\frac{q^{(r-m)k-1}}{k_q\alpha^{k_q}} \\ =& \sum_{i=1}^{n} q^{mi}\begin{bmatrix} n-i+m-1 \\ m-1\end{bmatrix}_q\sum_{k=1}^{i}\begin{bmatrix} i-k+r-m-1 \\ r-m-1\end{bmatrix}_q\frac{q^{(r-m)k-1}}{k_q\alpha^{k_q}} \\ =& \sum_{i=1}^{n} q^{m(n-i+1)}\begin{bmatrix} i+m-2 \\ m-1\end{bmatrix}_q\sum_{k=1}^{n-i+1}\begin{bmatrix} n-i+1+r-m-k-1 \\ r-m-1\end{bmatrix}_q\frac{q^{(r-m)k-1}}{k_q\alpha^{k_q}} \\ =& \sum_{i=1}^{n} q^{m(n-i+1)}\begin{bmatrix} i+m-2 \\ m-1\end{bmatrix}_q H_{n-i+1}^{r-m}(\alpha,q) \\ =& \sum_{i=0}^{n-1} q^{m(n-i)}\begin{bmatrix} i+m-1 \\ m-1\end{bmatrix}_q H_{n-i}^{r-m}(\alpha,q). \end{aligned}$$

Thus, the proof is complete. □

Theorem 2.8. *For $0 < m < r < n$, we have*

$$\sum_{k=1}^{m} q^{k(n-m+k)}\begin{bmatrix} m \\ k\end{bmatrix}_q H_{n-m+k}^{r-k}(\alpha,q)=H_n^r(\alpha,q)-H_{n-m}^r(\alpha,q).$$

Proof. By (2.5), we have

$$\begin{aligned}
&\sum_{k=1}^{m} q^{k(n-m+k)} \begin{bmatrix} m \\ k \end{bmatrix}_q H_{n-m+k}^{r-k}(\alpha, q) \\
=\ & \sum_{k=1}^{m} q^{k(n-m+k)} \begin{bmatrix} m \\ k \end{bmatrix}_q \sum_{i=1}^{n-m+k} \begin{bmatrix} n-m+r-i-1 \\ r-k-1 \end{bmatrix}_q \frac{q^{(r-k)i-1}}{i_q \alpha^{i_q}} \\
=\ & \sum_{k=1}^{m} q^{k(n-m+k)} \begin{bmatrix} m \\ k \end{bmatrix}_q \sum_{i=1}^{n-m} \begin{bmatrix} n-m+r-i-1 \\ r-k-1 \end{bmatrix}_q \frac{q^{(r-k)i-1}}{i_q \alpha^{i_q}} \\
& + \sum_{k=1}^{m} \sum_{i=n-m+1}^{n-m+k} q^{k(n-m+k-i)} \begin{bmatrix} m \\ k \end{bmatrix}_q \begin{bmatrix} n-m+r-i-1 \\ r-k-1 \end{bmatrix}_q \frac{q^{ri-1}}{i_q \alpha^{i_q}} \\
=\ & \sum_{i=1}^{n-m} \frac{q^{ri-1}}{i_q \alpha^{i_q}} \sum_{k=1}^{m} q^{k(n-m+k-i)} \begin{bmatrix} m \\ k \end{bmatrix}_q \begin{bmatrix} n-m+r-i-1 \\ r-k-1 \end{bmatrix}_q \\
& + \sum_{k=1}^{m} \sum_{i=1}^{k} q^{k(k-i)} \begin{bmatrix} m \\ k \end{bmatrix}_q \begin{bmatrix} r-i-1 \\ r-k-1 \end{bmatrix}_q \frac{q^{r(i+n-m)-1}}{(i+n-m)_q \alpha^{(i+n-m)_q}},
\end{aligned}$$

and applying some elementary operations, equals

$$\begin{aligned}
=\ & \sum_{i=1}^{n-m} \frac{q^{ri-1}}{i_q \alpha^{i_q}} \sum_{k=1}^{m} q^{k(n-m+k-i)} \begin{bmatrix} m \\ k \end{bmatrix}_q \begin{bmatrix} n-m+r-i-1 \\ r-k-1 \end{bmatrix}_q \\
& + \sum_{i=1}^{m} \sum_{k=i}^{m} q^{k(k-i)} \begin{bmatrix} m \\ k \end{bmatrix}_q \begin{bmatrix} r-i-1 \\ r-k-1 \end{bmatrix}_q \frac{q^{r(i+n-m)-1}}{(i+n-m)_q \alpha^{(i+n-m)_q}} \\
=\ & \sum_{i=1}^{n-m} \frac{q^{ri-1}}{i_q \alpha^{i_q}} \sum_{k=1}^{m} q^{k(n-m+k-i)} \begin{bmatrix} m \\ k \end{bmatrix}_q \begin{bmatrix} n-m+r-i-1 \\ r-k-1 \end{bmatrix}_q \\
& + \sum_{i=n-m+1}^{n} \sum_{k=i-n+m}^{m} q^{k(k-i+n-m)} \begin{bmatrix} m \\ k \end{bmatrix}_q \begin{bmatrix} r-i+n-m-1 \\ r-k-1 \end{bmatrix}_q \frac{q^{ri-1}}{i_q \alpha^{i_q}} \\
=\ & \sum_{i=1}^{n-m} \frac{q^{ri-1}}{i_q \alpha^{i_q}} \sum_{k=1}^{m} q^{k(n-m+k-i)} \begin{bmatrix} m \\ k \end{bmatrix}_q \begin{bmatrix} n-m+r-i-1 \\ r-k-1 \end{bmatrix}_q \\
& + \sum_{i=n-m+1}^{n} \frac{q^{ri-1}}{i_q \alpha^{i_q}} \sum_{k=0}^{n-i} q^{k(k+i-n+m)} \begin{bmatrix} m \\ k+i-n+m \end{bmatrix}_q \begin{bmatrix} r-i+n-m-1 \\ r-k-i+n-m-1 \end{bmatrix}_q \\
=\ & \sum_{i=1}^{n-m} \frac{q^{ri-1}}{i_q \alpha^{i_q}} \sum_{k=0}^{m} q^{k(n-m+k-i)} \begin{bmatrix} m \\ k \end{bmatrix}_q \begin{bmatrix} n-m+r-i-1 \\ r-k-1 \end{bmatrix}_q - \sum_{i=1}^{n-m} \frac{q^{ri-1}}{i_q \alpha^{i_q}} \begin{bmatrix} n-m+r-i-1 \\ r-1 \end{bmatrix}_q \\
& + \sum_{i=n-m+1}^{n} \frac{q^{ri-1}}{i_q \alpha^{i_q}} \sum_{k=0}^{n-i} q^{k(k+m-(n-i))} \begin{bmatrix} m \\ n-i-k \end{bmatrix}_q \begin{bmatrix} r-i+n-m-1 \\ k \end{bmatrix}_q.
\end{aligned}$$

With help of q-analog of Vandermonde identity (1.2), we can show that for $m \leq r-1$,

$$\begin{aligned}
&\sum_{k=1}^{m} q^{k(n-m+k)} \begin{bmatrix} m \\ k \end{bmatrix}_q H_{n-m+k}^{r-k}(\alpha, q) \\
=& \sum_{i=1}^{n-m} \begin{bmatrix} n+r-i-1 \\ r-1 \end{bmatrix}_q \frac{q^{ri-1}}{i_q \alpha^{i_q}} + \sum_{i=n-m+1}^{n} \begin{bmatrix} n+r-i-1 \\ r-1 \end{bmatrix}_q \frac{q^{ri-1}}{i_q \alpha^{i_q}} \\
& - \sum_{i=1}^{n-m} \begin{bmatrix} n-m+r-i-1 \\ r-1 \end{bmatrix}_q \frac{q^{ri-1}}{i_q \alpha^{i_q}} \\
=& \sum_{i=1}^{n} \begin{bmatrix} n+r-i-1 \\ r-1 \end{bmatrix}_q \frac{q^{ri-1}}{i_q \alpha^{i_q}} - \sum_{i=1}^{n-m} \begin{bmatrix} n-m+r-i-1 \\ r-1 \end{bmatrix}_q \frac{q^{ri-1}}{i_q \alpha^{i_q}} \\
=& H_n^r(\alpha, q) - H_{n-m}^r(\alpha, q),
\end{aligned}$$

as claimed. □

3. Some applications for the q-analog of the generalized hyperharmonic numbers of order r

In this section, we define an $n \times n$ matrix $A_n = [a_{i,j}]$ with $a_{i,j} = H_i^r(j, q)$ and an $n \times n$ matrix $C_n = [c_{i,j}]$ with $c_{i,j} = H_i^s(j, q)$.
Now we can give the following theorem:

Theorem 3.1. *For $n > 0$, we have*

$$A_n = F_n C_n, \tag{3.1}$$

where the $n \times n$ matrix $F_n = [f_{i,j}]$ with

$$f_{i,j} = \begin{cases} q^{j(r-s)} \begin{bmatrix} i-j+r-s-1 \\ i-j \end{bmatrix}_q & \text{if } i \geq j, \\ 0 & \text{otherwise,} \end{cases}$$

where $0 \leq s \leq r-1$.

Proof. It is clear that $a_{1,1} = H_1^r(1, q) = q^{r-1} = f_{1,1} c_{1,1}$. For $i = 1$, $j > 1$, we write

$$\begin{aligned}
&\sum_{k=1}^{n} f_{1,k} c_{k,j} \\
=& \sum_{k=1}^{n} q^{k(r-s)} \begin{bmatrix} r-s-k \\ 1-k \end{bmatrix}_q H_k^s(j, q) \\
=& q^{r-s} H_1^s(j, q) = q^{r-s} \frac{q^{s-1}}{j} = \frac{q^{r-1}}{j} = H_1^r(j, q) = a_{1,j}.
\end{aligned}$$

For $i > 1$ and $j \geq 1$, we obtain

$$\begin{aligned}
&\sum_{k=1}^{n} f_{i,k} c_{k,j} \\
=\ & \sum_{k=1}^{n} q^{k(r-s)} \begin{bmatrix} i+r-s-k-1 \\ i-k \end{bmatrix}_q H_k^s(j,q) \\
=\ & \sum_{k=1}^{i} q^{k(r-s)} \begin{bmatrix} i+r-s-k-1 \\ i-k \end{bmatrix}_q \sum_{t=1}^{k} \begin{bmatrix} k+s-t-1 \\ s-1 \end{bmatrix}_q \frac{q^{st-1}}{t_q j^{t_q}} \\
=\ & \sum_{t=1}^{i} \frac{q^{rt-1}}{t_q j^{t_q}} \sum_{k=t}^{i} q^{(k-t)(r-s)} \begin{bmatrix} i+r-s-k-1 \\ i-k \end{bmatrix}_q \begin{bmatrix} k+s-t-1 \\ s-1 \end{bmatrix}_q \\
=\ & \sum_{t=1}^{i} \frac{q^{rt-1}}{t_q j^{t_q}} \sum_{k=s-1}^{i-t+s-1} q^{(k-s+1)(r-s)} \begin{bmatrix} i+r-t-k-2 \\ r-s-1 \end{bmatrix}_q \begin{bmatrix} k \\ s-1 \end{bmatrix}_q .
\end{aligned}$$

Taking $s-1=p$, $i-t+r-2=n$ and $r-s-1=m$ in (1.1), we get

$$\sum_{t=1}^{i} \begin{bmatrix} i+r-t-1 \\ r-1 \end{bmatrix}_q \frac{q^{rt-1}}{t_q j^{t_q}} = H_i^r(j,q) = a_{i,j},$$

as claimed. □

Theorem 3.2. *For $n > 0$, the inverse of F_n has the terms as follows:*

$$f_{i,j}^{-1} = \begin{cases} (-1)^{i+j} q^{-i(r-s)+\binom{i-j}{2}} \begin{bmatrix} r-s \\ i-j \end{bmatrix}_q & \text{if } 0 \leq i-j \leq r-s+1, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. For $i=j$, we have

$$\begin{aligned}
&\sum_{k=1}^{n} f_{i,k}^{-1} f_{k,i} \\
=\ & \sum_{k=1}^{n} (-1)^{i+k} q^{-i(r-s)+\binom{i-k}{2}+i(r-s)} \begin{bmatrix} r-s \\ i-k \end{bmatrix}_q \begin{bmatrix} k-i+r-s-1 \\ k-i \end{bmatrix}_q \\
=\ & \sum_{k=1}^{n} (-1)^{i+k} q^{\binom{i-k}{2}} \begin{bmatrix} r-s \\ i-k \end{bmatrix}_q \begin{bmatrix} k-i+r-s-1 \\ k-i \end{bmatrix}_q = 1.
\end{aligned}$$

Similarly, for $i \neq j$, we have $\sum_{k=1}^{n} f_{i,k}^{-1} f_{k,j} = 0$. Thus, the proof is complete. □

Corollary 3.3. *For $1 \leq n \leq r-s+1$, then*

$$H_n^s(\alpha,q) = \sum_{k=1}^{n} (-1)^{n+k} q^{-n(r-s)+\binom{n-k}{2}} \begin{bmatrix} r-s \\ n-k \end{bmatrix}_q H_k^r(\alpha,q).$$

Proof. Equating (n,α)-entries of the matrix multiplication $F_n^{-1}A_n = C_n$ gives the claimed result. □

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